

# A REMARK ON A PRIORI ESTIMATE FOR THE NAVIER-STOKES EQUATIONS WITH THE CORIOLIS FORCE

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**ABSTRACT.** The Cauchy problem for the Navier-Stokes equations with the Coriolis force is considered. It is proved that a similar a priori estimate, which is derived for the Navier-Stokes equations by Lei and Lin [11], holds under the effect of the Coriolis force. As an application existence of a unique global solution for arbitrary speed of rotation is proved, as well as its asymptotic behavior.

## 1. INTRODUCTION

In this note, we consider the initial value problem of the Navier-Stokes equations with the Coriolis force in  $\mathbb{R}^3$ ,

$$(NS_\Omega) \quad \begin{cases} \partial_t u - \nu \Delta u + \Omega e_3 \times u + (u, \nabla)u + \nabla p = 0, & \text{in } (0, \infty) \times \mathbb{R}^3, \\ \operatorname{div} u = 0, & \text{in } (0, \infty) \times \mathbb{R}^3, \\ u|_{t=0} = u_0, & \text{in } \mathbb{R}^3, \end{cases}$$

where  $u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$  denotes the unknown velocity field, and  $p = p(t, x)$  denotes the unknown scalar pressure, while  $u_0 = u_0(x) = (u_0^1(x), u_0^2(x), u_0^3(x))$  denotes the initial velocity field. The constant  $\nu > 0$  denotes the viscosity coefficient of the fluid, and  $\Omega \in \mathbb{R}$  represents the speed of rotation around the vertical unit vector  $e_3 = (0, 0, 1)$ , which is called the Coriolis parameter.

Recently, this problem gained some attention due to its importance in applications to geophysical flows, see e.g. [12, 3]. Mathematically,  $(NS_\Omega)$  also have a interesting feature that there exists a global solution for arbitrary large data provided the speed of rotation  $\Omega$  is large enough, see e.g. [1, 3, 7]. There are another type of results which shows the existence of a global solution uniformly in  $\Omega$  provided the data is sufficiently small, see e.g. [4, 6, 10, 8]. The purpose of this note is, concerning to the latter, to relax the smallness condition of the data, based on the idea for the Navier-Stokes equations,  $\Omega = 0$  in  $(NS_\Omega)$ , by [11].

Before stating our main results, we give a definition of function spaces. For  $m \in \mathbb{R}$ , we define

$$\chi^m(\mathbb{R}^3) := \{f \in \mathcal{S}' \mid \widehat{f} \in L_{\text{loc}}^1, \|f\|_{\chi^m} := \int_{\mathbb{R}^3} |\xi|^m |\widehat{f}(\xi)| d\xi < \infty\}.$$

In particular, we only use spaces  $\chi^{-1}$ ,  $\chi^0$ , and  $\chi^1$  below, so we summarize elementary estimates concerning the spaces we will use later.

**Lemma 1.** (1) For  $s > 1/2$ ,  $\|f\|_{\chi^{-1}(\mathbb{R}^3)} \leq C \|f\|_{L^2}^{1-\frac{1}{2s}} \|f\|_{\dot{H}^s}^{\frac{1}{2s}}$ .

(2)  $\|f\|_{\chi^0} \leq \|f\|_{\chi^{-1}}^{1/2} \|f\|_{\chi^1}^{1/2}$ .

(3)  $\|\nabla f\|_{L^\infty} \leq \|f\|_{\chi^1}$ .

*Proof.* (1) We take  $R > 0$ , which is determined later, to divide the integral

$$\begin{aligned} \|f\|_{\chi^{-1}} &= \int_{|\xi| \leq R} |\xi|^{-1} |\widehat{f}(\xi)| d\xi + \int_{|\xi| > R} |\xi|^{-1} |\widehat{f}(\xi)| d\xi \\ &\leq \left( \int_{|\xi| \leq R} |\xi|^{-2} d\xi \right)^{1/2} \|f\|_{L^2} + \left( \int_{|\xi| > R} |\xi|^{-2-2s} d\xi \right)^{1/2} \|f\|_{\dot{H}^s} \\ &= |S^2|^{1/2} \left( R^{1/2} \|f\|_{L^2} + \frac{1}{\sqrt{2s-1}} R^{-s+1/2} \|f\|_{\dot{H}^s} \right). \end{aligned}$$

Then, choosing  $R = \|f\|_{L^2}^{-1/s} \|f\|_{\dot{H}^2}^{1/s}$ , we obtain the desired result.

(2) This estimate is easily derived by the Hölder inequality,

$$\|f\|_{\chi^0} = \int |\xi|^{-1/2} |\widehat{f}(\xi)|^{1/2} |\xi|^{1/2} |\widehat{f}(\xi)|^{1/2} d\xi \leq \|f\|_{\chi^{-1}}^{1/2} \|f\|_{\chi^1}^{1/2}.$$

(3) This is also easily derived from the Fourier inversion formula and the Hausdorff-Young inequality.  $\square$

Now we state our main results.

**Theorem 1.** Let  $\Omega \in \mathbb{R}$ , and let  $u_0 \in \chi^{-1}$  satisfy  $\operatorname{div} u_0 = 0$  and  $\|u_0\|_{\chi^{-1}} < (2\pi)^3 \nu$ . For  $T > 0$ , assume that  $u \in C([0, T]; \chi^{-1})$  is a solution to  $(\text{NS}_\Omega)$  in the distribution sense satisfying

$$u \in L^1(0, T; \chi^1), \quad \partial_t u \in L^1(0, T; \chi^{-1}).$$

Then,  $u$  satisfies

$$(1.1) \quad \|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3} \|u_0\|_{\chi^{-1}}) \int_0^t \|u(\tau)\|_{\chi^1} d\tau \leq \|u_0\|_{\chi^{-1}}, \quad 0 \leq t < T.$$

*Remark 2.* (1) This a priori estimate is first derived in the case  $\Omega = 0$  in [11, Proof of Theorem 1.1]. Here, Theorem 1 states that the same estimate also holds under the effect of the Coriolis force.

(2) In this note, we define the Fourier transform of  $f$  by

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) := \int e^{-ix \cdot \xi} f(x) dx.$$

The constant  $(2\pi)^3$  in the theorem appears from the following formula:

$$\mathcal{F}[fg](\xi) = (2\pi)^{-3} (\widehat{f} * \widehat{g})(\xi),$$

where  $f * g$  denotes the convolution of  $f$  and  $g$ .

(3) From the a priori estimate (1.1), we especially obtain

$$\|u\|_{L^\infty(0,T;\chi^{-1})} \leq \|u_0\|_{\chi^{-1}}, \quad \|u\|_{L^1(0,T;\chi^1)} \leq \frac{\|u_0\|_{\chi^{-1}}}{\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}}.$$

As an application of Theorem 1 we obtain a unique global solution to  $(NS_\Omega)$ .

**Theorem 2.** *Let  $\Omega \in \mathbb{R}$ . Assume that  $u_0 \in \chi^{-1}(\mathbb{R}^3)$  satisfy  $\operatorname{div} u_0 = 0$  and  $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$ . Then, there exists a unique global solution  $u \in C([0, \infty); \chi^{-1})$  to  $(NS_\Omega)$  satisfying*

$$u \in L^1(0, \infty; \chi^{-1}), \quad \partial_t u \in L^1_{\text{loc}}(0, \infty; \chi^{-1}),$$

and

$$\sup_{t>0} \{ \|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}) \int_0^t \|u(\tau)\|_{\chi^1} d\tau \} \leq \|u_0\|_{\chi^{-1}}.$$

*Remark 3.* (1) There are several results which treats the existence of a unique global solution to  $(NS_\Omega)$ , see [8] and reference therein. In particular, the spaces  $FM_0^{-1}$ , which is considered by Giga, Inui, Mahalov, and Saal [4], and  $\mathcal{B}_{1,2}^{-1}$  by [8], are larger than  $\chi^{-1}$ . However, the advantage of this result is that the condition of the size of the data is merely  $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$ .

(2) In the Navier-Stokes equations, the case  $\Omega = 0$ , the corresponding result is proved in [11, Theorem 1.1]. We notice that there is also the another approach by [13, Theorem 1.3]. In our forthcoming paper we will consider that approach for  $(NS_\Omega)$ .

As a byproduct, we also obtain the following.

**Theorem 3.** *Let  $s > 3/2$  and  $\Omega \in \mathbb{R}$ . Assume that  $u_0 \in H^s(\mathbb{R}^3)$  satisfy  $\operatorname{div} u_0 = 0$  and  $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$ . Then, there exists a unique global solution  $u \in C([0, \infty); H^s)$  to  $(NS_\Omega)$  satisfying*

$$u \in AC([0, \infty); H^{s-1}) \cap L^1_{\text{loc}}(0, \infty; H^{s+1})$$

and

$$\sup_{t>0} \{ \|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}) \int_0^t \|u(\tau)\|_{\chi^1} d\tau \} \leq \|u_0\|_{\chi^{-1}}.$$

*Remark 4.* Since  $s > 3/2$ , we have  $H^s \hookrightarrow \chi^{-1}$  by Lemma 1. The condition  $s > 3/2$  follows from the local well-posedness by Proposition 6 which we employ for the proof. For a interval  $I$  and a Banach space  $X$ ,  $AC(I; X)$  denotes the space of  $X$ -valued absolutely continuous functions.

Next theorem states the asymptotic behavior of a given global solution to  $(NS_\Omega)$  in the framework of Sobolev spaces.

**Theorem 4.** *Let  $s > 1/2$  and  $\Omega \in \mathbb{R}$ . Assume that  $u \in C([0, \infty); H^s(\mathbb{R}^3))$  is a global solution to  $(NS_\Omega)$  satisfying*

$$(1.2) \quad u \in AC([0, \infty); H^{s-1}(\mathbb{R}^3)) \cap L^1_{\text{loc}}(0, \infty; H^{s+1}(\mathbb{R}^3)).$$

*Then,  $\lim_{t \rightarrow \infty} \|u(t)\|_{\chi^{-1}} = 0$ .*

*Remark 5.* In the Navier-Stokes case  $\Omega = 0$ , this result corresponds to the result in [2]. In that result, the assumption is only  $u \in C([0, \infty); \chi^{-1})$  is a global solution. Compared with that result, additional assumptions (1.2) are imposed for the uniqueness of solutions.

As an application of Theorem 4 we obtain the following.

**Corollary 5.** *The global solution to  $(NS_\Omega)$  derived in Theorem 3 satisfies*

$$\lim_{t \rightarrow 0} \|u(t)\|_{\chi^{-1}} = 0.$$

This paper is organized as follows. In Section 2 we give a proof of Theorem 1. In Section 3 we prove Theorem 3 as an application of Theorem 1. In Section 4 we give a proof of Theorem 2 by using Theorem 1 and Theorem 3. In Section 5 we give a proof of Theorem 4.

## 2. PROOF OF THEOREM 1

In this section we give a proof of Theorem 1.

*Proof of Theorem 1.* By applying the Fourier transform to the equation, we have

$$\partial_t \widehat{u} + \nu |\xi|^2 \widehat{u} + \Omega e_3 \times \widehat{u} + \mathcal{F}[(u, \nabla)u] + i\xi \widehat{p} = 0.$$

Thus, we obtain

$$\begin{aligned} \partial_t |\widehat{u}|^2 &= 2\text{Re}(\partial_t \widehat{u} \cdot \overline{\widehat{u}}) \\ &= -2\nu |\xi|^2 |\widehat{u}|^2 - 2\Omega \text{Re}[(e_3 \times \widehat{u}) \cdot \overline{\widehat{u}}] - 2\text{Re}\{\mathcal{F}[(u, \nabla)u] \cdot \overline{\widehat{u}}\} - 2\text{Re}[(i\xi \widehat{p}) \cdot \overline{\widehat{u}}]. \end{aligned}$$

Here, since

$$(e_3 \times \widehat{u}) \cdot \overline{\widehat{u}} = -\widehat{u}_2 \overline{\widehat{u}_1} + \widehat{u}_1 \overline{\widehat{u}_2} = 2i \text{Im}[\widehat{u}_1 \overline{\widehat{u}_2}],$$

we observe that  $\operatorname{Re}[(e_3 \times \widehat{u}) \cdot \overline{\widehat{u}}] = 0$ . Also, we have  $(i\xi \widehat{p}) \cdot \overline{\widehat{u}} = 0$ , since  $\operatorname{div} u = 0$ . Moreover, we notice that

$$\begin{aligned} \mathcal{F}[(u, \nabla)u]_j(\xi) &= \sum_{k=1}^3 (2\pi)^{-3} \widehat{u}_k * \widehat{\partial_k u_j}(\xi) \\ &= \sum_{k=1}^3 (2\pi)^{-3} \int \widehat{u}_k(\xi - \eta) i\eta_k \widehat{u_j}(\eta) d\eta \\ &= \sum_{k=1}^3 (2\pi)^{-3} i\xi_k \int \widehat{u}_k(\xi - \eta) \widehat{u_j}(\eta) d\eta, \end{aligned}$$

since  $\sum_{k=1}^3 (\xi_k - \eta_k) \widehat{u}_k(\xi - \eta) = 0$ . Therefore, we obtain

$$\begin{aligned} \partial_t |\widehat{u}|^2 + 2\nu |\xi|^2 |\widehat{u}|^2 &\leq 2(2\pi)^{-3} \sum_{j,k=1}^3 |\xi_k| (|\widehat{u}_k| * |\widehat{u_j}|) |u_j| \\ &\leq 2(2\pi)^{-3} |\xi| |\widehat{u}| (|\widehat{u}| * |\widehat{u}|). \end{aligned}$$

Then, for  $\varepsilon > 0$ , we observe that

$$\begin{aligned} \partial_t (|\widehat{u}|^2 + \varepsilon)^{1/2} &= \frac{\partial_t |\widehat{u}|^2}{2(|\widehat{u}|^2 + \varepsilon)^{1/2}} \\ &\leq -\frac{\nu |\xi|^2 |\widehat{u}|^2}{(|\widehat{u}|^2 + \varepsilon)^{1/2}} + (2\pi)^{-3} \frac{|\xi| |\widehat{u}|}{(|\widehat{u}|^2 + \varepsilon)^{1/2}} (|\widehat{u}| * |\widehat{u}|). \end{aligned}$$

Integrating with respect to  $t$ , we obtain

$$\begin{aligned} (|\widehat{u}(t, \xi)|^2 + \varepsilon)^{1/2} &+ \int_0^t \frac{\nu |\xi|^2 |\widehat{u}(\tau, \xi)|^2}{(|\widehat{u}(\tau, \xi)|^2 + \varepsilon)^{1/2}} d\tau \\ &\leq (|\widehat{u}_0(\xi)|^2 + \varepsilon)^{1/2} + (2\pi)^{-3} \int_0^t \frac{|\xi| |\widehat{u}(\tau, \xi)|}{(|\widehat{u}(\tau, \xi)|^2 + \varepsilon)^{1/2}} (|\widehat{u}(\tau)| * |\widehat{u}(\tau)|)(\xi) d\tau. \end{aligned}$$

Then, letting  $\varepsilon \rightarrow 0$ , we get

$$|\widehat{u}(t, \xi)| + \int_0^t \nu |\xi|^2 |\widehat{u}(\tau, \xi)| d\tau \leq |\widehat{u}_0(\xi)| + (2\pi)^{-3} \int_0^t |\xi| (|\widehat{u}(\tau)| * |\widehat{u}(\tau)|)(\xi) d\tau.$$

Finally, dividing by  $|\xi|$ , and then integrating over  $\mathbb{R}^n$ , we obtain

$$\|u(t)\|_{\chi^{-1}} + \nu \int_0^t \|u(\tau)\|_{\chi^1} d\tau \leq \|u_0\|_{\chi^{-1}} + (2\pi)^{-3} \int_0^t \|u(\tau)\|_{\chi^0}^2 d\tau.$$

By applying Lemma 1 (2), we obtain,

$$(2.1) \quad \|u(t)\|_{\chi^{-1}} + \nu \|u\|_{L^1((0,t); \chi^1)} \leq \|u_0\|_{\chi^{-1}} + (2\pi)^{-3} \|u\|_{L^\infty((0,t); \chi^{-1})} \|u\|_{L^1((0,t); \chi^1)}.$$

To derive the desired estimate (1.1), it suffices to prove that

$$\|u\|_{L^\infty((0,t); \chi^{-1})} \leq \|u_0\|_{\chi^{-1}}, \quad 0 \leq t < T.$$

For the proof, we first show that

$$(2.2) \quad \|u(t)\|_{\chi^{-1}} < (2\pi)^3 \nu, \quad 0 \leq t < T$$

holds by contradiction. From the assumption  $\|u_0\|_{\chi^{-1}} < (2\pi)^3 \nu$  and  $u \in C([0, T]; \chi^{-1})$ , we observe that there exists  $\delta > 0$  such that (2.2) holds on  $[0, \delta)$ . Now assume that there exists  $t_0 \in (0, T)$  such that  $\|u(t)\|_{\chi^{-1}} < (2\pi)^3 \nu$  for  $0 < t < t_0$  and

$$\|u(t_0)\|_{\chi^{-1}} = (2\pi)^3 \nu,$$

then by (2.1) we reach the contradiction

$$(2\pi)^3 \nu = \|u(t_0)\|_{\chi^{-1}} \leq \|u_0\|_{\chi^{-1}} < (2\pi)^3 \nu,$$

since  $\|u\|_{L^\infty((0, t_0); \chi^{-1})} = (2\pi)^3 \nu$ . Therefore, we obtain (2.2). Finally, applying (2.2) to estimate on the right hand side of (2.1), we obtain

$$\|u(t)\|_{\chi^{-1}} < \|u_0\|_{\chi^{-1}}, \quad 0 \leq t < T.$$

This completes the proof.  $\square$

### 3. PROOF OF THEOREM 3

Below we fix  $\Omega \in \mathbb{R}$ . For the existence of local solutions, we employ the following result.

**Proposition 6.** *Let  $s > 3/2$ . For  $u_0 \in H^s(\mathbb{R}^3)$  with  $\operatorname{div} u_0 = 0$ , there exists  $T = T(|\Omega|, s, \|u_0\|_{H^s}) > 0$  such that  $(\text{NS}_\Omega)$  admits a unique strong solution  $u \in C([0, T]; H^s(\mathbb{R}^3))$  satisfying*

$$u \in AC([0, T]; H^{s-1}(\mathbb{R}^3)) \cap L^1(0, T; H^{s+1}(\mathbb{R}^3)).$$

*Remark 7.* (1) For the proof, we refer to [9, Lemma 3.1]. The idea is based on to construct the solution to the integral equation

$$u(t) = e^{\nu t \Delta} u_0 - \Omega \int_0^t e^{\nu(t-\tau) \Delta} \mathbb{P}(e_3 \times u)(\tau) d\tau - \int_0^t e^{\nu(t-\tau) \Delta} \mathbb{P}(u, \nabla u)(\tau) d\tau$$

by the contraction mapping argument, where  $\mathbb{P} = (\delta_{ij} + R_i R_j)_{i,j}$  is the Helmholtz projection. We notice that the condition in [9, Lemma 3.1] is  $s > 3/2 + 1$ , because their main subject is the Euler equation. For the above statement,  $s > 3/2$  is sufficient.

(2) In this proposition, the size of  $T$  is characterized by

$$(3.1) \quad C_0 |\Omega| T + C_1 \|u_0\|_{H^s} (T + T^{1/2} \nu^{-1/2}) \leq \frac{1}{2}.$$

(3) Since  $s > 3/2$ , the solution constructed by Proposition 6 satisfies the assumptions in Theorem 1. In particular, since

$$\partial_t u = \nu \Delta u - \Omega \mathbb{P}(e_3 \times u) - \mathbb{P}(u, \nabla u) \quad \text{in } H^{s-1}$$

holds for a.e.  $t \in (0, T)$ , we easily observe that  $\partial_t u \in L^1(0, T; \chi^{-1})$ .

We will use the following energy estimate.

**Proposition 8.** *Let  $s \geq 0$  and  $T > 0$ . Assume that  $u \in C([0, T]; H^s(\mathbb{R}^3))$  is a solution to  $(NS_\Omega)$  satisfying*

$$u \in AC([0, T]; H^{s-1}(\mathbb{R}^3)) \cap L^1(0, T; H^{s+1}(\mathbb{R}^3)).$$

*Then,  $u$  satisfies*

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} e^{C \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau}, \quad 0 \leq t < T.$$

*Remark 9.* For the proof of this proposition, we also refer to [9, Proof of Theorem 4.1]. There, we easily observe that

$$\frac{d}{dt} \|u(t)\|_{H^s} \leq C \|\nabla u(t)\|_{L^\infty} \|u(t)\|_{H^s}$$

holds for  $s \geq 0$ . We notice that the term concerning  $\Omega e_3 \times u$  vanishes due to

$$\Omega(e_3 \times u) \cdot u = 0.$$

Now we are in a position to prove Theorem 3.

*Proof of Theorem 3.* Let  $T^*$  be the maximal existence time of a unique solution derived by applying Proposition 6 repeatedly. Now assume  $T^* < \infty$ . Then, by (3.1), we must have

$$(3.2) \quad \lim_{t \rightarrow T^*} \|u(t)\|_{H^s} = \infty.$$

Since this solution satisfies the energy estimate in Proposition 8, we have

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} e^{C \int_0^{T^*} \|\nabla u(\tau)\|_{L^\infty} d\tau}, \quad 0 \leq t < T^*.$$

Then, since  $\|u_0\|_{\chi^{-1}} < (2\pi)^3 \nu$ , applying Theorem 1 we obtain

$$\int_0^{T^*} \|\nabla u(\tau)\|_{L^\infty} d\tau \leq \|u\|_{L^1(0, T^*; \chi^1)} \leq \frac{\|u_0\|_{\chi^{-1}}}{\nu - (2\pi)^{-3} \|u_0\|_{\chi^{-1}}}.$$

This implies  $\sup_{0 < t < T^*} \|u(t)\|_{H^s} < \infty$ , which contradicts to (3.2).  $\square$

#### 4. PROOF OF THEOREM 2

In this section we give a proof of Theorem 2.

For  $u_0 \in \chi^{-1}$  and  $R > 0$ , we set

$$D_R = \{\xi \in \mathbb{R}^3 \mid |\xi| \leq R, |\widehat{u}_0(\xi)| \leq R\}, \quad u_0^R = \mathcal{F}^1[\chi_{D_R} \widehat{u}_0],$$

where  $\chi_{D_R}$  denotes the characteristic function of  $D_R$ . Then, we observe that

$$u_0^R \in H^\infty, \quad \|u_0^R\|_{\chi^{-1}} \leq \|u_0\|_{\chi^{-1}},$$

and from Lebesgue's dominant convergence theorem,

$$(4.1) \quad \|u_0^R - u_0\|_{\chi^{-1}} = \int |\xi|^{-1} (\chi_{D_R}(\xi) - 1) |\widehat{u}_0(\xi)| d\xi \rightarrow 0, \quad R \rightarrow \infty,$$

since  $u_0 \in \chi^{-1}$ .

Now we apply Theorem 3 for the data  $u_0^R$  to derive a unique global solution  $u^R \in C([0, \infty); H^s)$  satisfying

$$u^R \in AC([0, \infty); H^{s-1}) \cap L^1(0, \infty; H^{s+1})$$

for  $s > 3/2$ , and

$$(4.2) \quad \|u^R\|_{L^\infty(0, \infty; \chi^{-1})} \leq \|u_0\|_{\chi^{-1}}, \quad \|u^R\|_{L^1(0, \infty; \chi^1)} \leq \frac{\|u_0\|_{\chi^{-1}}}{\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}}.$$

Below we first show that  $\{u^R\}$  is a Cauchy sequence in  $L^\infty(0, \infty; \chi^{-1})$ . If we set  $w = u^R - u^{R'}$ , then  $w$  satisfies

$$\partial_t w - \nu \Delta w + \Omega e_3 \times w + (u^R, \nabla)w + (w, \nabla)u^{R'} + \nabla(p^R - p^{R'}) = 0.$$

Then, from the argument in the proof of Theorem 1, we obtain

$$\begin{aligned} & \|w(t)\|_{\chi^{-1}} + \nu \int_0^t \|w(\tau)\|_{\chi^1} d\tau \\ & \leq \|w(0)\|_{\chi^{-1}} + (2\pi)^{-3} \int_0^t (\|u^R(\tau)\|_{\chi^0} + \|u^{R'}(\tau)\|_{\chi^0}) \|w(\tau)\|_{\chi^0} d\tau. \end{aligned}$$

Here, applying Lemma 1 (2) we have

$$(4.3) \quad \begin{aligned} \|u^R\|_{\chi^0} \|w\|_{\chi^0} & \leq \|u^R\|_{\chi^{-1}}^{1/2} \|u^R\|_{\chi^1}^{1/2} \|w\|_{\chi^{-1}}^{1/2} \|w\|_{\chi^1}^{1/2} \\ & \leq \frac{1}{2} (\|u^R\|_{\chi^{-1}} \|w\|_{\chi^1} + \|u^R\|_{\chi^1} \|w\|_{\chi^{-1}}). \end{aligned}$$

Therefore, combining (4.2) we obtain

$$(4.4) \quad \begin{aligned} & \|w(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}) \int_0^t \|w(\tau)\|_{\chi^1} d\tau \\ & \leq \|w(0)\|_{\chi^{-1}} + \int_0^t a(\tau) \|w(\tau)\|_{\chi^{-1}} d\tau, \end{aligned}$$

where

$$a(\tau) = \frac{1}{2(2\pi)^3} (\|u^R(\tau)\|_{\chi^1} + \|u^{R'}(\tau)\|_{\chi^1}).$$

Note that by (4.2) we have a uniform bound

$$\int_0^\infty a(\tau) d\tau \leq \frac{\|u_0\|_{\chi^{-1}}}{(2\pi)^3 \nu - \|u_0\|_{\chi^{-1}}}.$$

Thus, applying Gronwall's inequality to (4.4) we obtain

$$(4.5) \quad \|w(t)\|_{\chi^{-1}} \leq \|w(0)\|_{\chi^{-1}} e^{\int_0^t a(\tau) d\tau},$$



which implies

$$\|u^R - u^{R'}\|_{L^\infty(0,\infty;\chi^{-1})} \leq \|u_0^R - u_0^{R'}\|_{\chi^{-1}} e^{\int_0^\infty a(\tau) d\tau} \rightarrow 0, \quad R, R' \rightarrow \infty.$$

Therefore, there exists  $u \in L^\infty(0, \infty; \chi^{-1})$  such that  $u^R \rightarrow u$  in  $L^\infty(0, \infty; \chi^{-1})$ .

We next show the convergence in  $L^1(0, \infty; \chi^1)$ . The convergence in  $L^\infty(0, \infty; \chi^{-1})$  implies there exists a subsequence  $\{u^{\tilde{R}}\}$  such that for a.e.  $(t, \xi)$ ,

$$\mathcal{F}[u^{\tilde{R}}](t, \xi) \rightarrow \widehat{u}(t, \xi), \quad R \rightarrow \infty.$$

Therefore, by Fatou's lemma and the estimate derived from (4.4) and (4.5),

$$\|w\|_{L^1(0,\infty;\chi^1)} \leq \frac{\|w(0)\|_{\chi^{-1}}}{\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}} \left(1 + \int_0^\infty a(\tau) d\tau e^{\int_0^\infty a(\tau) d\tau}\right),$$

we conclude that

$$\|u^R - u\|_{L^1(0,\infty;\chi^1)} \leq \liminf_{\tilde{R} \rightarrow 0} \|u^R - u^{\tilde{R}}\|_{L^1(0,\infty;\chi^1)} \rightarrow 0, \quad R \rightarrow \infty.$$

From convergence in  $L^\infty(0, \infty; \chi^{-1}) \cap L^1(0, \infty; \chi^1)$  we observe that the limit  $u$  satisfies the integral equation

$$u(t) = e^{\nu t \Delta} u_0 - \Omega \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P}(e_3 \times u)(\tau) d\tau - \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(\tau) d\tau,$$

which  $u^R$  also satisfies for the data  $u_0^R$ . In fact, we are able to estimate

$$\begin{aligned} \|e^{\nu t \Delta} u_0^R - e^{\nu t \Delta} u_0\|_{\chi^{-1}} &\leq \|u_0^R - u_0\|_{\chi^{-1}}, \\ \left\| \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P}(e_3 \times u^R)(\tau) d\tau - \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P}(e_3 \times u)(\tau) d\tau \right\|_{\chi^{-1}} \\ &\leq t \|u^R - u\|_{L^\infty(0,\infty;\chi^{-1})}, \end{aligned}$$

and

$$\begin{aligned} &\left\| \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P}(u^R, \nabla u^R)(\tau) d\tau - \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P}(u, \nabla u)(\tau) d\tau \right\|_{\chi^{-1}} \\ &\leq \int_0^t (\|u^R(\tau)\|_{\chi^0} + \|u(\tau)\|_{\chi^0}) \|u^R(\tau) - u(\tau)\|_{\chi^0} d\tau \\ &\leq C(\|u^R - u\|_{L^\infty(0,\infty;\chi^{-1})} + \|u^R - u\|_{L^1(0,\infty;\chi^1)}), \end{aligned}$$

where we applied the estimate like (4.3) and the uniform bound (4.2).

We next show  $\partial_t u \in L^1(0, T; \chi^{-1})$  for any  $T > 0$ , which implies  $u \in C([0, \infty); \chi^{-1})$ . To prove this, we consider to apply  $\partial_t$  to the right hand of the integral equation. We first

notice that for the first term

$$\begin{aligned}
\|\partial_t e^{\nu \Delta t} u_0\|_{L^1(0, \infty; \chi^{-1})} &= \|\nu \Delta e^{\nu \Delta t} u_0\|_{L^1(0, \infty; \chi^{-1})} \\
&= \nu \int_0^\infty \int |\xi| e^{-\nu |\xi|^2 t} |\widehat{u}_0(\xi)| d\xi dt \\
&= \int |\xi|^{-1} |\widehat{u}_0(\xi)| d\xi = \|u_0\|_{\chi^{-1}}
\end{aligned}$$

holds by changing the order of the integrals. This type of argument can be found in [10, Lemma 3.5]. (See also [5, Theorem 2.5] in relation with the  $L^1$ -maximal regularity.) So, it suffices to show that  $\partial_t \Phi \in L^1(0, T; \chi^{-1})$ , where

$$\Phi(t) = \Omega \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P}(e_3 \times u)(\tau) d\tau + \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(\tau) d\tau.$$

Since

$$\partial_t \Phi(t) = \Delta \Phi(t) + \mathbb{P}(e_3 \times u)(t) + \mathbb{P} \nabla \cdot (u \otimes u)(t),$$

we will check each term on the right hand side belongs to  $L^1(0, T; \chi^{-1})$ . It is easy to see that

$$\begin{aligned}
\int_0^T \|\mathbb{P}(e_3 \times u)\|_{\chi^{-1}} dt &\leq T \|u\|_{L^\infty(0, T; \chi^{-1})}, \\
\int_0^T \|\mathbb{P} \nabla \cdot (u \otimes u)(t)\|_{\chi^{-1}} dt &\leq \int_0^T \|u\|_{\chi^0}^2 d\tau \leq \|u\|_{L^\infty(0, T; \chi^{-1})} \|u\|_{L^1(0, T; \chi^1)}, \\
\int_0^T \left\| \Omega \int_0^t \Delta e^{\nu(t-\tau)\Delta} \mathbb{P}(e_3 \times u)(\tau) d\tau \right\|_{\chi^{-1}} dt &\leq |\Omega| T \|u\|_{L^1(0, T; \chi^1)}.
\end{aligned}$$

And applying the argument the above again,

$$\begin{aligned}
&\int_0^T \left\| \int_0^t \Delta e^{\nu(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(\tau) d\tau \right\|_{\chi^{-1}} dt \\
&\leq \int_0^T \left( \int_0^t \int |\xi|^2 e^{-\nu(t-\tau)|\xi|^2} (|\widehat{u}(\tau)| * |\widehat{u}(\tau)|)(\xi) d\xi d\tau \right) dt \\
&= \int_0^T \int \left( \int_\tau^T |\xi|^2 e^{-\nu(t-\tau)|\xi|^2} dt \right) (|\widehat{u}(\tau)| * |\widehat{u}(\tau)|)(\xi) d\xi d\tau \\
&\leq \int_0^T \|u\|_{\chi^0}^2 d\tau \leq \|u\|_{L^\infty(0, T; \chi^{-1})} \|u\|_{L^1(0, T; \chi^1)}.
\end{aligned}$$

Finally, we notice that (4.5) implies the uniqueness of solutions.

## 5. PROOF OF THEOREM 4

In this section we give a proof of Theorem 4.

We take  $\varepsilon > 0$  arbitrary small. Since  $u_0 \in H^s \hookrightarrow \chi^{-1}$ , we are able to choose  $R_0 > 0$  such that

$$\int_{|\xi| > R_0} |\xi|^{-1} |\widehat{u}_0(\xi)| d\xi < \frac{\varepsilon}{2}.$$

Now we set

$$v_0 = \mathcal{F}^{-1}[\chi_{\{|\xi| \leq R_0\}} \widehat{u}_0], \quad w_0 = \mathcal{F}^{-1}[\chi_{\{|\xi| > R_0\}} \widehat{u}_0].$$

Then, we observe that  $v_0 \in H^\infty$ ,  $w_0 \in H^s$ ,  $u_0 = v_0 + w_0$ , and

$$\|w_0\|_{\chi^{-1}} < \frac{\varepsilon}{2}.$$

By applying Theorem 3 for the initial data  $w_0$  we obtain the solution  $(w, p_w)$  to  $(NS_\Omega)$ . Then,  $w \in C([0, \infty); H^s) \cap L^1(0, \infty; H^{s+1})$  satisfies

$$(5.1) \quad \|w(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3} \|w_0\|_{\chi^{-1}}) \int_0^t \|w(\tau)\|_{\chi^1} d\tau \leq \|w_0\|_{\chi^{-1}} < \frac{\varepsilon}{2}, \quad t > 0.$$

Now we set  $v := u - w$ . Then,  $v \in C([0, \infty); H^s)$  satisfies

$$v \in AC([0, \infty); H^{s-1}(\mathbb{R}^3)) \cap L^1(0, \infty; H^{s+1}(\mathbb{R}^3))$$

and

$$\begin{cases} \partial_t v + \nu \Delta v + \Omega e_3 \times v + (v, \nabla)v + (w, \nabla)v + (v, \nabla)w + \nabla(p - p_w) = 0, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0. \end{cases}$$

Taking  $L^2$ -inner product with  $v$ , the equation becomes

$$\frac{d}{dt} \|v(t)\|_{L^2}^2 + \nu \|\nabla v(t)\|_{L^2}^2 = \langle (v, \nabla)w, v \rangle_{L^2}.$$

Since

$$\langle (v, \nabla)w, v \rangle_{L^2} = -\langle w, (v, \nabla)v \rangle_{L^2},$$

we obtain

$$\begin{aligned} |\langle (v, \nabla)w, v \rangle_{L^2}| &\leq \|w\|_{L^\infty} \|v\|_{L^2} \|\nabla v\|_{L^2} \\ &\leq C \|w\|_{\chi^0} \|v\|_{L^2} \|\nabla v\|_{L^2} \\ &\leq C_\nu \|w\|_{\chi^0}^2 \|v\|_{L^2}^2 + \frac{\nu}{2} \|\nabla v\|_{L^2}^2 \end{aligned}$$

Therefore, we obtain

$$\frac{d}{dt} \|v(t)\|_{L^2}^2 + \frac{\nu}{2} \|\nabla v(t)\|_{L^2}^2 = C_\nu \|w(t)\|_{\chi^0}^2 \|v(t)\|_{L^2}^2.$$

Then, by Gronwall's inequality,

$$(5.2) \quad \|v(t)\|_{L^2}^2 + \frac{\nu}{2} \int_0^t \|\nabla v(\tau)\|_{L^2}^2 d\tau \leq \|v(0)\|_{L^2}^2 e^{C_\nu \int_0^t \|w(\tau)\|_{\chi^0}^2 d\tau}.$$

Here, by (5.1) we have

$$(5.3) \quad \int_0^t \|w(\tau)\|_{\chi^0}^2 d\tau \leq \|w\|_{L^\infty((0,t);\chi^{-1})} \|w\|_{L^1((0,t);\chi^1)} \leq \frac{\|w_0\|_{\chi^{-1}}^2}{\nu - (2\pi)^{-3} \|w_0\|_{\chi^{-1}}}.$$

Therefore, by Lemma 1 (1), (5.2), (5.3) we obtain

$$\int_0^\infty \|v(t)\|_{\chi^{-1}}^4 d\tau \leq \int_0^\infty \|v(t)\|_{L^2}^2 \|\nabla v(t)\|_{L^2}^2 d\tau \leq \frac{2}{\nu} \|v_0\|_{L^2}^4 \exp\left(\frac{C_\nu \|w_0\|_{\chi^{-1}}^2}{\nu - (2\pi)^{-3} \|w_0\|_{\chi^{-1}}}\right).$$

Since  $v \in C([0, \infty); \chi^{-1})$ , we observe that there exists  $t_0 > 0$  such that  $\|v(t_0)\|_{\chi^{-1}} < \varepsilon/2$ , and thus we have  $\|u(t_0)\|_{\chi^{-1}} \leq \|v(t_0)\|_{\chi^{-1}} + \|w(t_0)\|_{\chi^{-1}} < \varepsilon$ . So, applying Theorem 3 for the data  $u(t_0)$  we obtain

$$\|u(t)\|_{\chi^{-1}} \leq \|u(t_0)\|_{\chi^{-1}} < \varepsilon, \quad t > t_0,$$

which implies  $\lim_{t \rightarrow 0} \|u(t)\|_{\chi^{-1}} = 0$ .

Here, we notice that in the final part of the proof we need the uniqueness of solutions, which is assured in our class of solutions. In fact, if  $u_1$ , and  $u_2 \in C([0, \infty); H^s)$  are two solutions to (NS<sub>Ω</sub>) satisfying

$$u_1, u_2 \in AC([0, \infty); H^{s-1}(\mathbb{R}^3)) \cap L_{\text{loc}}^1(0, \infty; H^{s+1}(\mathbb{R}^3)),$$

then,  $\tilde{u} := u_1 - u_2$  satisfies  $\operatorname{div} \tilde{u} = 0$  and

$$\partial_t \tilde{u} + \nu \Delta \tilde{u} + \Omega e_3 \times \tilde{u} + (\tilde{u}, \nabla) \tilde{u} + (u_1, \nabla) \tilde{u} + (\tilde{u}, \nabla) u_2 + \nabla(p_1 - p_2) = 0,$$

and thus we obtain

$$\frac{d}{dt} \|\tilde{u}(t)\|_{L^2}^2 + \frac{\nu}{2} \|\nabla \tilde{u}(t)\|_{L^2}^2 = |\langle (\tilde{u}, \nabla) u_2, \tilde{u} \rangle_{L^2}| \leq \|\nabla u_2(t)\|_{L^\infty} \|\tilde{u}(t)\|_{L^2}^2.$$

Therefore, we have

$$\frac{d}{dt} \|\tilde{u}(t)\|_{L^2}^2 = C \|u_2(t)\|_{H^{s+1}} \|\tilde{u}(t)\|_{L^2}^2$$

and Gronwall's inequality implies  $\tilde{u}(t) = 0$  for  $t > 0$ .

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